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FAST TRACK COMMUNICATION

Markov chain of distances between parked cars**Petr Seba**^{1,2,3}¹ University of Hradec Králové, Hradec Králové, Czech Republic² Institute of Physics, Academy of Sciences of the Czech Republic, Prague, Czech Republic³ Doppler Institute for Mathematical Physics and Applied Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Prague, Czech Republic

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Online at stacks.iop.org/JPhysA/41/122003**Abstract**

We describe the distribution of distances between parked cars as a solution of certain Markov processes and show that its solution is obtained with the help of a distributional fixed point equation. Under certain conditions the process is solved explicitly. The resulting probability density is compared with the actual parking data measured in the city.

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We focus on the spacing distribution between cars parked parallel to the curb somewhere in the city center. We will assume that the street is long enough to enable a parallel parking of many cars. Moreover we assume that there are no driveways or side streets in the segment of interest and that the street is free of any kind of marked parking lots or park meters. So the drivers are free to park the car anywhere provided they find an empty space to do it. Finally we assume that many cars are cruising for parking. So there are no free parking lots and a car can park only when another car leaves the street.

The standard way to describe random parking is the continuous version of the random sequential adsorption model known also as the ‘random car parking problem’—see [1, 2] for review. In this model it is assumed that all cars have the same length l_0 and park on randomly chosen places. Once parked the cars do not leave the street. This model leads to predictions that can be easily verified. First of all it gives a relation between the mean bumper-to-bumper distance \bar{D} and the car length: $\bar{D} \sim 0.337l_0$. Further the probability density $Q(D)$ of the car distances D behaves like [3–6]

$$Q(D) \approx -\ln(D). \quad (1)$$

To test this results real parking data were collected recently in the center of London [7]. The average distance between the parked cars was 152 cm which fit perfectly with the relation $\bar{D} \sim 0.337l_0$ for $l_0 = 450$ cm. The gap density (1) was however fully incompatible with the observed facts. The relation (1) gives $Q(D) \rightarrow \infty$ for $D \rightarrow 0$. The data from London showed however that in reality $Q(D) \rightarrow 0$ as $D \rightarrow 0$. Rawal and Rogers [7] used

therefore an amended version of the model with a car re-positioning process (describing the car manoeuvring) to fit the data. Later Abul-Magd [8] pointed out that the car spacings observed in London can be well described by the Gaussian unitary ensemble of random matrices [9, 10] if the parked cars are regarded as particles of a one-dimensional interacting gas. In both cases the authors assumed that $Q(D) \sim D^2$ for small D . However the origin of the particular car re-positioning or car interaction remained unclear.

Our aim here is to show that the car parking can be understood as a simple Markov process. The main difference to the previous approach is that we enable the parked cars to leave the street vacating the space for a new car to park there. The distance distribution is obtained as a steady solution of the repeated car parking and car leaving process.

To derive the equations we describe a car parking on a roundabout junction (we know that it is not allowed to park there, but it simplifies the consideration). The idea is simple: assume that all cars have the same length l_0 and park on a roundabout junction with a circumference L . The length L is chosen in interval $3l_0 < L < 4l_0$ so that maximally three cars can park there. These cars define three spacings D_1, D_2, D_3 with the constrain

$$D_1 + D_2 + D_3 + 3l_0 = L. \quad (2)$$

The parking process goes as follows: one randomly chosen car leaves the street and the two adjoining lots merge into a single one. In the second step a new car parks into the empty space and splits it again into two smaller lots. Such fragmentation and coagulation processes were discussed intensively since they apply for instance to the computer memory allocation—see [11] for review.

If the car leaves and the neighboring spacings—say the spacings D_1, D_2 —merge into a single lot D we get

$$D = D_1 + D_2 + l_0. \quad (3)$$

When a new car parks to D it splits it again into two lots \tilde{D}_1, \tilde{D}_2 :

$$\tilde{D}_1 = a(D - l_0) \quad \tilde{D}_2 = (1 - a)(D - l_0), \quad (4)$$

where $a \in (0, 1)$ is a random variable with a probability density $P(a)$. The distribution $P(a)$ describes the parking preference of the driver. We assume that all drivers have identical preferences, i.e. identical $P(a)$. (The meaning of the variable a is straightforward. For $a = 0$ the car parks immediately in front of the car delimiting the parking lot from the left without leaving any empty space (very unworthy way to park). For $a = 1/2$ it parks directly to the center of the lot D and for $a = 1$ it stops exactly behind the car on the right.) Combining (3) and (4) lead to

$$\tilde{D}_1 = a(D_1 + D_2) \quad \tilde{D}_2 = (1 - a)(D_1 + D_2) \quad (5)$$

and the car length l_0 drops out. Taking the first of these equations and using the relation (2) we finally obtain

$$\tilde{D}_1 = a(L - 3l_0 - D_3). \quad (6)$$

The pair D_1, D_2 is in no way particular. The same works when dealing with an arbitrary pair of D_k, D_l ; $k \neq l$; $k, l = 1, 2, 3$. The last step is now easy. Suppose that after many steps a steady state is reached. Then the distributions of D_k , $k = 1, 2, 3$ are identical and the distances D_k can be regarded as copies of a random variable D . The equation (6) becomes

$$D_{n+1} = a_n(L - 3l_0 - D_n), \quad (7)$$

where D_n denotes the variable D after n steps and a_n are independent identically distributed random variables with a common probability density $P(a)$. So D_n , $n = 1, 2, 3, \dots$ represents

a simple Markov chain. We are interested in the solution of (7) in the limit $n \rightarrow \infty$. One can show easily [12] that the limiting variable D_∞ solves the equation $D_\infty \triangleq a(L - 3l_0 - D_\infty)$. The symbol \triangleq means that left and right-hand sides of the equation have identical probability distributions. The constant $L - 3l_0$ can be scaled out and set to 1. The final probability density $Q(D)$ of D solves the equation

$$D \triangleq a(1 - D). \quad (8)$$

Distributional fixed point equations of this type are mathematically well studied—see for instance [13–15] although not much is known about their exact solutions. They are used for instance in the insurance and financial mathematics and represent the value of a commitment to make regular payments [16]. (In the actuarial mathematics such equations are called perpetuities.) But they arise also in the relation with recursive algorithms such as the selection algorithm Quickselect, see, e.g. [17, 18] and in many other areas.

For us the key point is the choice of the probability distribution $P(a)$ characterizing the parking behavior of the driver. It is reasonable to use a distribution with $P(0) = P(1) = 0$. A natural candidate is the β distribution

$$P(a) = \beta(\alpha_1, \alpha_2, a) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} a^{\alpha_1-1} (1-a)^{\alpha_2-1} \quad (9)$$

with $\alpha_1, \alpha_2 > 1$. Independent β distributed random variables fulfil a couple relations that will be helpful for us. [19, 20]. In particular the following holds:

Statement 1. Let x_1, x_2 be independent random variables. Further let x_1 be $\beta(\alpha_1, \alpha_1 + \alpha_2, x)$ and x_2 be $\beta(\alpha_3, \alpha_2, x)$ distributed. Then the variable $x_3 := x_2 - x_2 x_1$ has a $\beta(\alpha_3, \alpha_1 + \alpha_2, x)$ distribution for all $\alpha_1, \alpha_2, \alpha_3 > 0$.

Taking $c = a$ we see that the variables x_1 and x_3 are equally distributed. In other words the above statement leads to a solution of equation (8) as follows:

Statement 2. Let $P(a)$ be given by (9). Then the probability density $Q(D)$ solving equation (8) is given by $Q(D) = \beta(\alpha_1, \alpha_1 + \alpha_2, D)$.

For parking it is natural to assume a symmetric $P(a)$: $P(a) = P(1 - a)$, i.e. the drivers are not biased to park more closely to a car adjacent from the behind or from the front. A β distribution is symmetric if $\alpha_1 = \alpha_2$. So we will take $P(a) = \beta(\alpha, \alpha, a)$ which leads to the solution of (8): $Q(D) = \beta(\alpha, 2\alpha, D)$. The parameter α is free and can be in principle fitted to describe the parking data. However the behavior of $Q(D)$ for small D display the capability of the driver to exploit small distances during the parking maneuver. It is related to the general human ability to estimate small distances since the collision avoidance during parking is guided visually. Assuming this virtue to be shared by all drivers we finally come to the conclusion that the behavior of $Q(D)$ for small D should be generic, i.e. independent on the particular city or situation.

These arguments eventually fix the parameter α since $Q(D) = \beta(\alpha, 2\alpha, D) \sim D^{\alpha-1}$ as $D \rightarrow 0$. The observations from London [7, 8] show that $Q(D) \approx D^2$ for small D leading to $\alpha = 3$.

Rescaling finally the variable D to achieve a mean distance equal to 1 we get $Q(D) = (1/3)\beta(3, 6, D/3)$. Please note that the distribution is fixed (by the assumption that its behavior for small D is generic) and does not contain any free parameter.

To compare this prediction with actual parking data we measured the bumper-to-bumper distances between cars parked on two different streets in the center of Hradec Kralove (Czech Republic). Both streets were long enough and usually without any free parking places. So a new car was able to park there only when another car left the street. In addition the mean

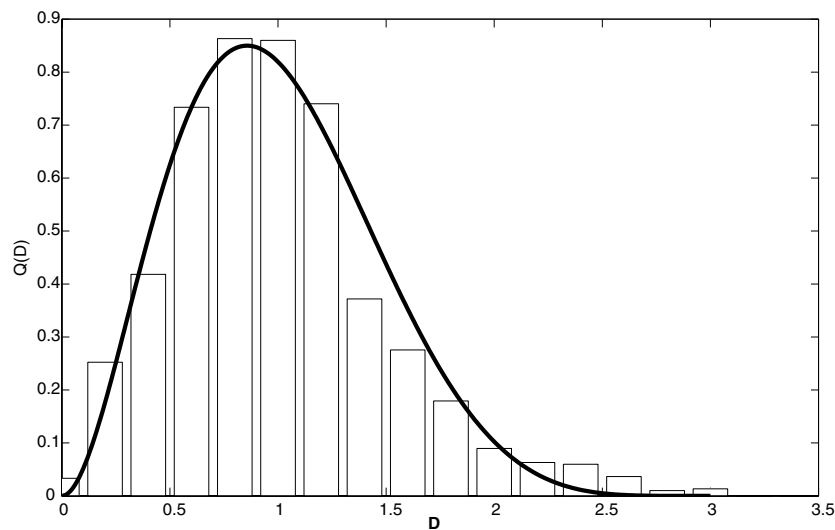


Figure 1. The measured distance distribution for cars parking in the center of Hradec Kralove is compared with the solution of equation (8) with a distribution $P(a) = \beta(3, 3, a)$. The measured data were scaled to a mean distance equal to 1.

parking time was quite short so the parked cars exchanged frequently during the day. We measured overall 1200 car spacings. The measured mean car distance was 140 cm. It is slightly less than the value 154 cm reported in [7], but we have smaller cars in the Czech Republic.

The result is plotted in figure 1 and the agreement with the prediction is reasonable. It has to be notified that the distribution $Q(D) = (1/3)\beta(3, 6, D/3)$ is quite close to the result obtained for the Gaussian unitary ensemble and used in [8]. But $Q(D) = 0$ for $D > 3$. This means that in the model there are no parking lots larger than three. For the actual data (due to the scaling to $\bar{D} = 1$) this means that there are no lots longer than 420 cm. This is really a favorable property, since such large lots are immediately occupied by a new car and do not persist. Such feature is however not present in the random matrix description [8] since this distribution has a long tail.

The presented model is a simplification. In reality the cars are not parking on roundabout junctions and their lengths are not equal. But nevertheless the simplified and solvable model leads to reasonably good agreement with the data.

To summarize: we have shown that car parking can be well described by a simple Markov process. The limiting distribution of this process is solvable and leads to a prediction of the car distance distribution which is in agreement with the actual data measured on the streets of the city Hradec Kralove.

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